Computational Plasma Physics
Research Notes

Alberto J. Garcia

Contents

1 Charged-Particle Motion **1**
1.1 Primary Goal . . . 1
1.2 Kinematics of a Charged Particle . . . 1
1.2.1 Case 1: No Magnetic Field . . . 2
1.2.2 Case 2: No Electric Field . . . 4
1.2.3 Energy conservation . . . 8
1.2.4 Velocity components . . . 9
1.3 Computational Methods . . . 11
1.4 Results . . . 16
1.5 Conceptual Questions . . . 22
1.6 Summary . . . 24

A Levi-Civita symbol **26**

References **27**

Chapter 1

Charged-Particle Motion

1.1 Primary Goal

The primary goal of this week is to numerically solve the Lorentz force equation for a moving charged particle [1]. We will be answering the following question:

How accurately do different numerical integrators reproduce charged-particle motion in a magnetic field?

1.2 Kinematics of a Charged Particle

The force on a charged particle in an electromagnetic field is described by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1.1}$$

where $\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$. The following is a list of the variables:

- q - The charge of the particle.
- m - The mass of the particle.
- $\mathbf{E}$ - The electric field applied to the charge.
- $\mathbf{B}$ - The magnetic field applied to the charge.
- $\mathbf{v}$ - The velocity of the particle.
- $\mathbf{F}$ - The force applied on the particle.

To understand how a particle behaves different conditions, we will solve for the general solutions under two different limiting cases:

1. $\mathbf{B} \to 0, \mathbf{E} \neq 0$,
2. $\mathbf{E} \to 0, \mathbf{B} \neq 0$.

Writing Eq. (1.1) using Einstein notation and rearranging,

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{q}{m}(E_i + \varepsilon_{ijk} v_j B_k). \tag{1.2}$$

Assuming $\mathbf{E}$ and $\mathbf{B}$ are constant, we take the derivative with respect to time

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \frac{q}{m}\Big(\frac{\mathrm{d}E_i}{\mathrm{d}t} + \varepsilon_{ijk}\frac{\mathrm{d}v_j}{\mathrm{d}t} B_k\Big).$$

Since $\mathbf{E}$ is constant, $\frac{\mathrm{d}E_i}{\mathrm{d}t} = 0$, which yields

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \frac{q}{m}\varepsilon_{ijk}\frac{\mathrm{d}v_j}{\mathrm{d}t} B_k.$$

We can substitute Eq. (1.2) for the derivative term

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \Big(\frac{q}{m}\Big)^2 \varepsilon_{ijk}\Big(E_j + \epsilon_{jmn} v_m B_n\Big) B_k,$$
$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \Big(\frac{q}{m}\Big)^2 \Big[\varepsilon_{ijk} E_j B_k + \varepsilon_{ijk}\epsilon_{jmn} v_m B_n B_k\Big].$$

Using Eq. (A.3) in Appx. A, the combination of Levi-Civita symbols can be written as

$$\varepsilon_{ijk}\varepsilon_{jmn} = -\varepsilon_{jik}\varepsilon_{jmn} = -\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}. \tag{1.3}$$

Substituting these into the equation gives

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \Big(\frac{q}{m}\Big)^2 \Big[\varepsilon_{ijk} E_j B_k + (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) v_m B_n B_k\Big].$$

This gives us two terms:

$$\delta_{in}\delta_{km} v_m B_n B_k = (v_j B_j) B_i,$$
$$\delta_{im}\delta_{kn} v_m B_n B_k = v_i B_k B_k.$$

The purely magnetic field terms now become

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \Big(\frac{q}{m}\Big)^2 \Big[\varepsilon_{ijk} E_j B_k + (v_j B_j) B_i - v_i B_k B_k\Big]. \tag{1.4}$$

1.2.1 Case 1: No Magnetic Field

Let's consider the case where $\mathbf{B} \to 0, \mathbf{E} \neq 0$.

$$\lim_{\mathbf{B}\to 0} \frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = 0. \tag{1.5}$$

We integrate both sides twice

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = A \longrightarrow v_i = At + B.$$

In order to determine the coefficients we impose the boundary condition

$$v_i(0) = v_{i0},$$

as well as compare both derivative equations, $\frac{\mathrm{d}v_i}{\mathrm{d}t}\Big|_{t=0}$. For the first condition,

$$v_i(0) = B = v_{i0}.$$

For the second condition,

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}\Big|_{t=0} = A.$$

Referring back to Eq. (1.2), we obtain

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}\Big|_{t=0} = \frac{q}{m}E_i,$$

thus

$$A = \frac{q}{m}E_i.$$

Combining both and substituting back into v_i,

$$v_i = \frac{q}{m}E_i t + v_{i0}. \tag{1.6}$$

This is the general velocity equation of a charged particle in an electric field.

Writing out the vector components,

$$v_x(t) = \frac{qE_x}{m}t + v_{x0}, \quad v_y(t) = \frac{qE_y}{m}t + v_{y0}, \quad v_z(t) = \frac{qE_z}{m}t + v_{z0}. \tag{1.7}$$

These equations can be written as a vector equation,

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \frac{q}{m}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} t + \begin{bmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{bmatrix}, \tag{1.8}$$

or as

$$\mathbf{v} = \frac{q}{m}\mathbf{E}t + \mathbf{v}_0. \tag{1.9}$$

The position equation can be derived by integrating Eq. (1.9),

$$\mathbf{x} = \frac{q}{2m}\mathbf{E}t^2 + \mathbf{v}_0 t + \mathbf{x}_0, \tag{1.10}$$

where we have used the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

1.2.2 Case 2: No Electric Field

Let's now look at the case where $\mathbf{E} \to 0, \mathbf{B} \neq 0$. Referring back to Eq. (1.4) and removing the electric field term gives

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \left(\frac{q}{m}\right)^2 \Big[(v_j B_j) B_i - v_i B_k B_k\Big]. \tag{1.11}$$

Rearranging the terms and introducing the parameter Ω_c gives us

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} + \Omega_c^2 v_i = \left(\frac{q}{m}\right)^2 (v_j B_j) B_i. \tag{1.12}$$

where Ω_c is the cyclotron frequency defined by

$$\Omega_c^2 = \left(\frac{q}{m}\right)^2 B_k B_k, \tag{1.13}$$

with period

$$T_c = \frac{2\pi}{\Omega_c}. \tag{1.14}$$

We need to figure out if the r.h.s is constant or time-dependent. To do this, let's go back to Eq. (1.2),

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{q}{m} \varepsilon_{ijk} v_j B_k. \tag{1.15}$$

Multiplying both sides by B_i,

$$B_i \frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{q}{m} \varepsilon_{ijk} B_i v_j B_k.$$

Let's consider the multiplication

$$A = \varepsilon_{ijk} B_i v_j B_k.$$

We can flip the terms since multiplication is commutative

$$A = v_j \varepsilon_{ijk} B_i B_k. \tag{1.16}$$

We know that the product $B_i B_k$ is symmetric under exchange of indices $B_i B_k = B_k B_i$, so we can exchange the indices in Eq. (1.16)

$$A = v_j \epsilon_{kji} B_k B_i.$$

Using the permutation property of the Levi-Civita symbol, we see that

$$\epsilon_{kji} = -\varepsilon_{ijk},$$

thus

$$A = -\varepsilon_{ijk} v_j B_i B_k. \tag{1.17}$$

Summing Eq. (1.16) and Eq. (1.17)

$$2A = 0 \to A = 0$$

which means

$$B_i \frac{\mathrm{d}v_i}{\mathrm{d}t} = 0. \tag{1.18}$$

Let's look at the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(B_i v_i) = B_i \frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{\mathrm{d}B_i}{\mathrm{d}t} v_i.$$

We have shown that the first term is zero and the second term must be zero since $\mathbf{B}$ is constant. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(B_i v_i) = 0 \longrightarrow B_i v_i = H,$$

where H is a constant. We can determine what H is by using initial conditions

$$H = B_i v_i(0) = B_i v_{i0},$$

thus

$$H = v_{i0} B_i.$$

Plugging this into Eq. (1.12)

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} + \Omega_c^2 v_i = \left(\frac{q}{m}\right)^2 v_{j0} B_j B_i. \tag{1.19}$$

This is a second order differential equation composed of two parts: the homogenous and particular solution. We write it as

$$v_i(t) = v_i^{(h)}(t) + v_i^{(p)}(t).$$

To solve for the homogenous solution, we first write the characteristic equation using the ansatz $v_i^{(h)}(t) = e^{rt}$ and set the equation equal to zero,

$$e^{rt}(r^2 + \Omega_c^2) = 0$$

Since $e^{rt} \neq 0$, the only solutions are $r = \pm i\Omega_c$. Plugging these back into the ansatz and forming a linear combination of the solutions gives

$$v_i^{(h)}(t) = A\cos(\Omega_c t) + B\sin(\Omega_c t)$$

To solve for the particular solution, we use the ansatz $v_i^{(p)} = K$, where K is a constant. Since K is constant, only the term with no derivative survives,

$$v_i^{(p)} = \left(\frac{q}{m\Omega_c}\right)^2 v_{j0} B_j B_i.$$

Substituting for the cyclotron frequency, the full solution becomes

$$v_i(t) = A\cos(\Omega_c t) + B\sin(\Omega_c t) + v_{j0}\frac{B_j B_i}{B_k B_k}.$$

In order to determine the coefficients we impose the boundary condition

$$v_i(0) = v_{i0},$$

as well as compare both derivative equations at $t = 0$. For the first condition,

$$v_i(0) = A + v_{j0}\frac{B_j B_i}{B_k B_k} = v_{i0} \longrightarrow A = v_{i0} - v_{j0}\frac{B_j B_i}{B_k B_k}.$$

For the second condition,

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}\Big|_{t=0} = B\Omega_c.$$

Referring back to Eq. (1.2), we obtain

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}\Big|_{t=0} = \frac{q}{m}\varepsilon_{ijk}v_{j0}B_k.$$

Setting these two equal to each other gives

$$B = \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{j0}B_k.$$

We can now substitute in the coefficients and rearrange the terms,

$$v_i(t) = v_{i0}\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{j0}B_k\sin(\Omega_c t) + \frac{q^2}{m^2\Omega_c^2}v_{j0}B_j B_i\Big[1 - \cos(\Omega_c t)\Big]. \tag{1.20}$$

This is the general velocity equation of a charged particle in a magnetic field.

Let's now write the first component of Eq. (1.20),

$$v_1(t) = v_{10}\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{1jk}v_{j0}B_k\sin(\Omega_c t) + \frac{q^2}{m^2\Omega_c^2}v_{j0}B_j B_1\Big[1 - \cos(\Omega_c t)\Big]. \tag{1.21}$$

Let's expand the indices of the different expressions. For the first term we get

$$\varepsilon_{1jk}v_{j0}B_k = \varepsilon_{123}v_{20}B_3 + \varepsilon_{132}v_{30}B_2 = v_{20}B_3 - v_{30}B_2 \longrightarrow v_{y0}B_z - v_{z0}B_y$$

For the next term we get

$$v_{j0}B_j = v_{10}B_1 + v_{20}B_2 + v_{30}B_3 \longrightarrow \mathbf{v}_0 \cdot \mathbf{B}$$

Now, for the cyclotron frequency

$$\Omega_c^2 = \left(\frac{q}{m}\right)^2 B_k B_k = \left(\frac{q}{m}\right)^2 (B_1 \cdot B_1 + B_2 \cdot B_2 + B_3 \cdot B_3)$$
$$\Omega_c = \frac{q}{m}\sqrt{B_x^2 + B_y^2 + B_z^2}$$

Plugging the first two expanded terms into $v_1(t)$ and recognizing that $i = 1 \rightarrow x$ gives

$$v_x(t) = v_{x0}\cos(\Omega_c t) + \frac{q}{m\Omega_c}(v_{y0}B_z - v_{z0}B_y)\sin(\Omega_c t) + \frac{q^2}{m^2\Omega_c^2}(\mathbf{v}_0 \cdot \mathbf{B})B_x\Big[1 - \cos(\Omega_c t)\Big].$$

Let's examine how this equation behaves when there is a magnetic field along the z-axis, i.e., $\mathbf{B} = (0, 0, B_0)$. Substituting for Ω_c in the denominator of the second term, the equation reduces to

$$v_x = v_{x0}\cos(\Omega_c t) + v_{y0}\sin(\Omega_c t). \tag{1.22}$$

We can also work out what v_y and v_z are by following the same procedure. In this case,

$$v_y = v_{y0}\cos(\Omega_c t) - v_{x0}\sin(\Omega_c t), \tag{1.23}$$
$$v_z = v_{z0}. \tag{1.24}$$

These equations can be written in matrix form as

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \cos(\Omega_c t) & \sin(\Omega_c t) & 0 \\ -\sin(\Omega_c t) & \cos(\Omega_c t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{bmatrix}, \tag{1.25}$$

or as a matrix equation,

$$\mathbf{v} = R_z(\Omega_c t)\mathbf{v}_0, \tag{1.26}$$

where $R_z(\theta)$ describes a rotation about the z-axis. Note that while the rotation matrix used is for a clockwise rotation, the handedness of the rotation is actually defined by the sign on the charge located in the cyclotron frequency, Ω_c. For a negatively charged particle such as an electron, the rotation would be counterclockwise.

The position equation can be derived by integrating Eq. (1.20),

$$\begin{aligned} x_i(t) = x_{i0} &+ \left(\frac{v_{i0}}{\Omega_c} - \frac{q^2}{m^2\Omega_c^2} v_{j0} B_j B_i\right)\sin(\Omega_c t) \\ &+ \frac{q}{m\Omega_c^2}\varepsilon_{ijk} v_{j0} B_k \Big[1 - \cos(\Omega_c t)\Big] + \frac{q^2}{m^2\Omega_c} v_{j0} B_j B_i t, \end{aligned} \tag{1.27}$$

where we have used the initial condition $x_i(0) = x_{i0}$. For the case where $\mathbf{B} = (0, 0, B_0)$, the position equations are

$$x = x_0 + \frac{v_{x0}}{\Omega_c}\sin(\Omega_c t) + \frac{v_{y0}}{\Omega_c}\Big[1 - \cos(\Omega_c t)\Big], \tag{1.28}$$
$$y = y_0 + \frac{v_{y0}}{\Omega_c}\sin(\Omega_c t) - \frac{v_{x0}}{\Omega_c}\Big[1 - \cos(\Omega_c t)\Big], \tag{1.29}$$
$$z = z_0 + v_{z0} t. \tag{1.30}$$

These equations can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\Omega_c}\sin(\Omega_c t) & \frac{1}{\Omega_c}[1 - \cos(\Omega_c t)] & 0 \\ -\frac{1}{\Omega_c}[1 - \cos(\Omega_c t)] & \frac{1}{\Omega_c}\sin(\Omega_c t) & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \tag{1.31}$$

or as a matrix equation,

$$\mathbf{x} = \Pi(\Omega_c t)\mathbf{v}_0 + \mathbf{x}_0. \tag{1.32}$$

These equations describe a helical motion along the xy-axis with a linear drift along the z-axis. We can calculate the radius of the helical trajectory by writing the equations for x and y as a circle

$$x - x_0 - \frac{v_{y0}}{\Omega_c} = \frac{v_{x0}}{\Omega_c}\sin(\Omega_c t) - \frac{v_{y0}}{\Omega_c}\cos(\Omega_c t), \tag{1.33}$$

$$y - y_0 + \frac{v_{x0}}{\Omega_c} = \frac{v_{y0}}{\Omega_c}\sin(\Omega_c t) + \frac{v_{x0}}{\Omega_c}\cos(\Omega_c t). \tag{1.34}$$

Now, define $x_c = x_0 + \frac{v_{y0}}{\Omega_c}$ and $y_c = y_0 - \frac{v_{x0}}{\Omega_c}$,

$$x - x_c = \frac{v_{x0}}{\Omega_c}\sin(\Omega_c t) - \frac{v_{y0}}{\Omega_c}\cos(\Omega_c t), \tag{1.35}$$

$$y - y_c = \frac{v_{y0}}{\Omega_c}\sin(\Omega_c t) + \frac{v_{x0}}{\Omega_c}\cos(\Omega_c t). \tag{1.36}$$

We can now write these equations in the form of an equation of a circle centered at (x_c, y_c),

$$r^2 = (x - x_c)^2 + (y - y_c)^2.$$

The radius would then be

$$\rho = \sqrt{\left[\frac{v_{x0}}{\Omega_c}\sin(\Omega_c t) - \frac{v_{y0}}{\Omega_c}\cos(\Omega_c t)\right]^2 + \left[\frac{v_{y0}}{\Omega_c}\sin(\Omega_c t) + \frac{v_{x0}}{\Omega_c}\cos(\Omega_c t)\right]^2}.$$

After some simplification, we get

$$\rho_L = \frac{m}{|q|B_0}\sqrt{v_{x0}^2 + v_{y0}^2}. \tag{1.37}$$

This is known as the Larmor or cyclotron radius. A general solution can be derived from Eq. (1.27). I'll just state it here for completion

$$\rho_L = \frac{m}{|q|B_0}\sqrt{v_{i0}v_{i0} - \frac{(v_{j0}B_j)^2}{B_k B_k}}. \tag{1.38}$$

1.2.3 Energy conservation

To determine if the energy is conserved, we need to check that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = 0.$$

where K is the kinetic energy and U is the potential energy. In this case, all the energy is kinetic so $U = 0$. We can then differentiate the kinetic energy,

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}(v_i v_i) = \frac{1}{2}m\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}v_i + v_i\frac{\mathrm{d}v_i}{\mathrm{d}t}\right) = mv_i\frac{\mathrm{d}v_i}{\mathrm{d}t}.$$

Plugging Eq. (1.1) into the equation above

$$\frac{\mathrm{d}E}{\mathrm{d}t} = qv_i(E_i + \varepsilon_{ijk}v_jB_k).$$

Using a very similar process to obtain Eq. (1.18), we can show that $\varepsilon_{ijk}v_iv_jB_k = 0$ by exchanging the indices $i \leftrightarrow j$. The equation then becomes

$$\frac{\mathrm{d}K}{\mathrm{d}t} = qv_iE_i. \tag{1.39}$$

This equation tells us that the energy of a moving charged particle is conserved only if $\mathbf{E} = 0$. If $\mathbf{E} \neq 0$, the electric field is constantly transferring or removing energy from the particle, leading to the energy not being conserved.

1.2.4 Velocity components

We can also write the velocity in terms of the components parallel and perpendicular to the magnetic field. Let's rewrite Eq. (1.20) and substitute for the cyclotron frequency on the last term,

$$v_i(t) = v_{i0}\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{j0}B_k\sin(\Omega_c t) + \frac{v_{j0}B_j}{B_kB_k}B_i\Big[1 - \cos(\Omega_c t)\Big]. \tag{1.40}$$

The parallel velocity is defined as

$$v_\parallel(t) = v_i(t)\cdot\hat{b} = \frac{v_i(t)B_i}{B}, \tag{1.41}$$

which is described as the velocity projected onto the magnetic field. We take the dot product of Eq. (1.40) with B_i and noting that we've already shown the second term to be zero in Sec. 1.2.2,

$$v_i(t)B_i = v_{i0}B_i\cos(\Omega_c t) + \frac{v_{j0}B_j}{B_kB_k}B_iB_i\Big[1 - \cos(\Omega_c t)\Big].$$

This can be further simplied to

$$v_i(t)B_i = v_{i0}B_i\cos(\Omega_c t) + v_{j0}B_j\Big[1 - \cos(\Omega_c t)\Big]$$

by cancelling the dot product terms. Since the combinations of v_0 and B are dot products, we can relabel the dummy indices, which gives us

$$v_i(t)B_i = v_{j0}B_j.$$

From Eq. (1.41) we can show that

$$v_\parallel(t) = \frac{v_{j0}B_j}{B} = v_{\parallel 0}, \tag{1.42}$$

which shows that $v_\parallel$ is constant and cannot be changed by a uniform magnetic field. Its vector form can be written as

$$v_{\parallel i}(t) = \frac{v_{j0}B_j}{B^2}B_i. \tag{1.43}$$

The perpendicular velocity is defined as

$$v_{\perp i}(t) = v_i(t) - v_{\perp i}(t). \tag{1.44}$$

This gives the following general form for $v_\perp$

$$v_{\perp i}(t) = \left[v_{i0} - \frac{v_{j0}B_jB_i}{B^2}\right]\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{j0}B_k\sin(\Omega_c t).$$

Evaluating the solution using the same initial conditions as in Sec. 1.2.2,

$$v_{\perp i}(0) = v_{i0} - \frac{v_{j0}B_jB_i}{B^2} = v_{\perp i0}. \tag{1.45}$$

The equation then takes the form

$$v_{\perp i}(t) = v_{\perp i0}\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{j0}B_k\sin(\Omega_c t).$$

The second term can also be written in terms of these velocities. First, we recognize that

$$v_{j0} = v_{\perp j0} + v_{\parallel j0}.$$

Now, we can write

$$\begin{aligned}\varepsilon_{ijk}v_{j0}B_k &= \varepsilon_{ijk}(v_{\perp j0} + v_{\parallel j0})B_k,\\ &= \varepsilon_{ijk}v_{\perp j0}B_k + \varepsilon_{ijk}v_{\parallel j0}B_k.\end{aligned}$$

The second term will introduce a B_j term through $v_{\parallel j0}$, which we've already shown is zero, i.e., $\varepsilon_{ijk}B_jB_k = 0$. Thus,

$$v_{\perp i}(t) = v_{\perp i0}\cos(\Omega_c t) + \frac{q}{m\Omega_c}\varepsilon_{ijk}v_{\perp j0}B_k\sin(\Omega_c t). \tag{1.46}$$

We can also show that $v_\perp$ is constant by using Eq. (1.44)

$$v_{\perp i} = v_i - v_{\parallel i} \longrightarrow v_{\perp i}^2 = v_i^2 - v_{\parallel i}^2,$$

since the cross terms are zero due to the components being orthogonal. In Sec. 1.2.3, we showed that $\frac{\mathrm{d}K}{\mathrm{d}t} = 0$ when there is no electric field. Plugging in the equation for kinetic energy shows that,

$$\frac{\mathrm{d}}{\mathrm{d}t}(v_iv_i) = 0 \longrightarrow v_i^2 = \text{constant},$$

therefore,

$$v_\perp^2(t) = v_{i0}v_{i0} - \frac{(v_{j0}B_j)^2}{B_kB_k}, \tag{1.47}$$

$$|v_\perp(t)| = \sqrt{v_{i0}v_{i0} - \frac{(v_{j0}B_j)^2}{B_kB_k}} = v_{\perp 0}. \tag{1.48}$$

The magnitude of the velocity can also be calculated using Eq (1.42) and Eq. (1.45),

$$|v_i| = \sqrt{v_{i0}v_{i0} - \frac{(v_j B_j)^2}{B_k B_k}} + \frac{v_{j0}B_j}{B}. \tag{1.49}$$

Looking back at the definition of the Larmor radius we derived at the end of Sec. 1.2.2, we see that the expression in the square root is exactly $v_\perp^2(t)$. We can now write the Larmor radius as

$$\rho_L = \frac{v_{\perp 0}}{\Omega_c}. \tag{1.50}$$

1.3 Computational Methods

In order to solve the Lorentz force equation numerically, we must implement an integrator. Here, I will go over a few options:

Forward Euler Method
This is a first-order method and is considered to be the most basic method for numerical integration of differential equations. Since it is first-order, the truncation error is second-order. We can derive the method by applying a Taylor expansion of $y(t)$ about t_n

$$y(t_n + \delta t) \approx y(t_n) + \delta t y'(t_n) + \mathcal{O}(\delta t^2).$$

Defining $y_{n+1} = y(t_n + \delta t)$ and substituting in the general form of a first-order differential equation, $y'(t_n) = f(t_n, y_n)$ yields

$$y_{n+1} \approx y(t_n) + \delta t f(t_n, y_n).$$

This approximation has a first-order global error since it's linear, which is why it's known as a first-order method.

Let's understand how the energy behaves with this integrator. Referring back at Eq. (1.2) and assuming $\mathbf{E} = 0$ and $\mathbf{B}$ is constant, we can write

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \frac{q\Delta t}{m}(\mathbf{v}_n \times \mathbf{B}). \tag{1.51}$$

The change in kinetic energy between timesteps can also be written as

$$K_{n+1} - K_n = \frac{1}{2}m\Big(\|\mathbf{v}_{n+1}\|^2 - \|\mathbf{v}_n\|^2\Big). \tag{1.52}$$

Then, the $\mathbf{v}_{n+1}$ term becomes

$$\|\mathbf{v}_{n+1}\|^2 = \|\mathbf{v}_n\|^2 + \Big(\frac{q\Delta t}{m}\Big)^2 \|\mathbf{v}_n \times \mathbf{B}\|^2, \tag{1.53}$$

where the cross term cancels out since the dot product is zero. Now, Eq. (1.52) becomes

$$K_{n+1} - K_n = \frac{1}{2}\frac{q^2\Delta t^2}{m}\|\mathbf{v}_n \times \mathbf{B}\|^2 \geq 0. \tag{1.54}$$

From Eq. (1.54), we conclude that the kinetic energy will always increase with each consecutive time step. This is not a physical property, but a numerical property brought on by the intergrator we chose. We can also determine the overall shape of the kinetic energy over time. To do this, we split the velocity into the parallel and perpendicular components and plug them into the kinetic energy

$$K_n = \frac{1}{2}m\Big(\|\mathbf{v}_{\parallel,n}\|^2 + \|\mathbf{v}_{\perp,n}\|^2\Big), \tag{1.55}$$

where the cross terms cancel since $\mathbf{v}_{\parallel,n} \cdot \mathbf{v}_{\perp,n} = 0$. We can now focus on the $\mathbf{v}_{\perp,n}$ terms since the parallel terms conserve energy throughout the trajectory. Thus,

$$K_{\perp,n} = \frac{1}{2}m\|\mathbf{v}_{\perp,n}\|^2, \tag{1.56}$$

We can also write Eq. (1.53) in terms of the components,

$$\|\mathbf{v}_{\parallel,n+1} + \mathbf{v}_{\perp,n+1}\|^2 = \big(\mathbf{v}_{\parallel,n} + \mathbf{v}_{\perp,n}\big) + \Big(\frac{q\Delta t}{m}\Big)^2 \|\big(\mathbf{v}_{\parallel,n} + \mathbf{v}_{\perp,n}\big) \times \mathbf{B}\|^2.$$

Cancelling out the dot product between velocities and cross product between $\mathbf{v}_{\parallel,n}$ and $\mathbf{B}$, and substituting for the cyclotron frequency, we obtain

$$\begin{aligned}
\|\mathbf{v}_{\parallel,n+1}\|^2 &= \|\mathbf{v}_{\parallel,n}\|^2, \\
\|\mathbf{v}_{\perp,n+1}\|^2 &= \Big(1 + \Delta t^2\omega_c^2\Big)\|\mathbf{v}_{\perp,n}\|^2.
\end{aligned}$$

As with Eq. (1.54), these equations also show that the kinetic energy has to increase. Focusing on the perpendicular term, by taking consequtive timesteps, we can show that the following relationship holds:

$$\|\mathbf{v}_{\perp,n+1}\|^2 = \Big(1 + \Delta t^2\omega_c^2\Big)^{t/\Delta t+1}\|\mathbf{v}_{\perp,0}\|^2.$$

With this equation and the kinetic energy equation, we can write

$$K_{\perp,n} = \Big(1 + \Delta t^2\omega_c^2\Big)^{t/\Delta t} K_{\perp,0},$$

where we have shifted n by one without loss of generality. Taking the natural log of both sides yields

$$\ln\left(\frac{K_{\perp,n}}{K_{\perp,0}}\right) = \frac{t}{\Delta t}\ln\Big(1 + \Delta t^2\omega_c^2\Big).$$

If $\Delta t^2\omega_c^2 \ll 1$, we can expand the natural log,

$$\ln\left(\frac{K_{\perp,n}}{K_{\perp,0}}\right) = \frac{t}{\Delta t}\Big(\Delta t^2\omega_c^2\Big),$$

and thus

$$K_{\perp,n} \approx K_{\perp,0}\exp^{t\Delta t\omega_c^2}. \tag{1.57}$$

From Eq. (1.57), we can draw the conclusion that as time goes on, the kinetic energy calculated from Euler method solutions will increase exponentially. Likewise, for small timesteps, the kinetic energy will increase linearly. This tells us that while the Euler method may provide decent solutions for small values of Δt and short time periods, the kinetic energy will greatly increase as larger values of Δt are chosen and/or for large time periods, which will result in an increasing inaccuracy in the Forward Euler method solutions.

Runge-Kutta Method

The Runge-Kutta Method is one of the most well-known algorithms used to solve ordinary differential equation. The most common version of the Runge-Kutta method is the fourth-order Runge-Kutta (a.k.a. RK4). For comparisons, the Forward Euler Method is a first-order Runge-Kutta method. I'll only list the method here.

The implementation of the RK4 is as follows:

For $y' = f(y)$,

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{1.58}$$

where

$$\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f\Big(y_n + \frac{\Delta t}{2}k_1\Big), \\ k_3 &= f\Big(y_n + \frac{\Delta t}{2}k_2\Big), \\ k_4 &= f(y_n + \Delta t k_3). \end{aligned}$$

Just like with the Forward Euler method, we can show how the energy behaves when utilizing this integrator. We write Eq. (1.2) as

$$A\mathbf{v} = \frac{q}{m}\mathbf{v} \times \mathbf{B}, \tag{1.59}$$

where A is a linear operator, thus

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = A\mathbf{v}.$$

Here, $A^T = -A$ due to the property of the cross product. Let's write out the RK4. Since we assume $\mathbf{E} = 0$ and $\mathbf{B}$ is constant, the different k values take the following form:

$$\begin{aligned} k_1 &= A\mathbf{v}_n, \\ k_2 &= A\mathbf{v}_n + A^2\frac{\Delta t}{2}\mathbf{v}_n, \\ k_3 &= A\mathbf{v}_n + A^2\frac{\Delta t}{2}\mathbf{v}_n + A^3\frac{\Delta t^2}{4}\mathbf{v}_n, \\ k_4 &= A\mathbf{v}_n + A^2\frac{\Delta t}{2}\mathbf{v}_n + A^3\frac{\Delta t^2}{4}\mathbf{v}_n + A^4\frac{\Delta t^3}{4}\mathbf{v}_n. \end{aligned}$$

Plugging these into Eq. (1.58) and simplifying, we obtain

$$\mathbf{v}_{n+1} = \Big(\mathbf{I} + \Delta t A + \frac{\Delta t^2}{2!}A^2 + \frac{\Delta t^3}{3!}A^3 + \frac{\Delta t^4}{4!}A^4\Big)\mathbf{v}_n. \tag{1.60}$$

For simplicity, we can write Eq. (1.60) as

$$\mathbf{v}_{n+1} = \Lambda \mathbf{v}_n, \tag{1.61}$$

where Λ is a linear operator. We now take Eq. (1.52) and substitute Eq. (1.61), which yields

$$K_{n+1} - K_n = \frac{1}{2}m\mathbf{v}_n^T\big(\Lambda^T\Lambda - \mathbf{I}\big)\mathbf{v}_n.$$

We can work out the distribution of $\Lambda^T\Lambda$ by using the fact that $A^T = -A$. This works out to be

$$\Lambda^T\Lambda - \mathbf{I} = \frac{\Delta t^6}{72}A^6 + \frac{\Delta t^8}{576}A^8,$$

which then gives

$$K_{n+1} - K_n = \frac{1}{2}m\mathbf{v}_n^T\Big(\frac{\Delta t^6}{72}A^6 + \frac{\Delta t^8}{576}A^8\Big)\mathbf{v}_n. \tag{1.62}$$

Now, we must work out what the different powers of A are. Going back to Eq. (1.59), we multiply both sides by A,

$$\begin{aligned} A(A\mathbf{v}) &= \frac{q}{m}(A\mathbf{v}) \times \mathbf{B} = \frac{q}{m}\Big[\frac{q}{m}\mathbf{v} \times \mathbf{B}\Big] \times \mathbf{B}, \\ &= \frac{q^2}{m^2}\Big[\mathbf{B}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}(\mathbf{B}\cdot\mathbf{B})\Big]. \end{aligned}$$

We multiply both sides by A once more,

$$\begin{aligned} A(A^2\mathbf{v}) &= \frac{q^3}{m^3}\Big[\mathbf{B}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}(\mathbf{B}\cdot\mathbf{B})\Big] \times \mathbf{B}, \\ &= \frac{q^3}{m^3}\Big[(\mathbf{v}\cdot\mathbf{B})(\mathbf{B}\times\mathbf{B}) - (\mathbf{B}\cdot\mathbf{B})(\mathbf{v}\times\mathbf{B})\Big], \\ &= -\frac{q^3}{m^3}B^2(\mathbf{v}\times\mathbf{B}), \\ &= -\frac{q^2B^2}{m^2}\Big[\frac{q}{m}(\mathbf{v}\times\mathbf{B})\Big]. \end{aligned}$$

Substituting in for the cyclotron frequency and recognizing that we're obtained Eq. (1.59) once again, we can write

$$A^3\mathbf{v} = -\omega_c^2 A\mathbf{v}.$$

This relationship implies that

$$A^8\mathbf{v} = -\omega_c^2 A^6\mathbf{v}$$

since we can multiply both sides by A^5. Eq. (1.62) then becomes

$$K_{n+1} - K_n = \frac{m\Delta t^6}{144}\Big(1 - \frac{\omega_c^2\Delta t^2}{8}\Big)\mathbf{v}_n^T A^6\mathbf{v}_n. \tag{1.63}$$

The combination of $\mathbf{v}$ and A can be written as

$$\mathbf{v}_n^T A^6 \mathbf{v}_n = -(A^3\mathbf{v}_n)^T(A^3\mathbf{v}_n) = -\|A^3\mathbf{v}_n\|^2 = \omega_c^4 \left\| \frac{q}{m}(\mathbf{v}_n \times \mathbf{B}) \right\|^2,$$

thus,

$$K_{n+1} - K_n = -\frac{m\omega_c^4 \Delta t^6}{144}\Big(1 - \frac{\omega_c^2 \Delta t^2}{8}\Big) \left\| \frac{q}{m}(\mathbf{v}_n \times \mathbf{B}) \right\|^2.$$

We can now separate out the velocity in terms of parallel and perpendicular components. Since the parallel component crossed with the magnetic field is zero, we are left with

$$K_{n+1} - K_n = -\frac{m\omega_c^6 \Delta t^6}{144}\Big(1 - \frac{\omega_c^2 \Delta t^2}{8}\Big) \|\mathbf{v}_{\perp,n}\|^2. \tag{1.64}$$

From Eq. (1.64), we can consider multiple cases:

- Case 1: $\frac{\omega_c^2 \Delta t^2}{8} = 1$,
- Case 2: $\frac{\omega_c^2 \Delta t^2}{8} > 1$,
- Case 3: $\frac{\omega_c^2 \Delta t^2}{8} < 1$.

For case 1, Eq. (1.64) becomes $K_{n+1} = K_n$ and we obtain conservation of energy. Note that while the kinetic energy is conserved, this does not mean that the trajectory is correct since the direction of the velocity is not accounted for in Eq. (1.64). Another way for the system to conserve energy would be if $\mathbf{v}_{\perp,n} = 0$. This would mean that the particle only has parallel velocity, which corresponds to a constant planar velocity (hence conservation of energy) as shown by Eq. (1.42) and in Sec. 1.2.3.

For case 2, we would get an increasing kinetic energy as time passes. As with the Forward Euler method, this is not a physical property, but a numerical property brought on by the intergrator we chose. The RK4 solution is unstable due to either a large cyclotron frequency or a large timestep, which introduces a small error in the form of a numerical increase in the perpendicular velocity at every timestep. For both scenarios, the timestep has to be adjusted to correctly account for the motion caused by the magnetic field, thus moving the system towards case 1 or 3. If the timestep is too large, then the gyration will not be properly accounted for, causing the RK4 to provide an inadequate solution.

For case 3, we obtain a decreasing kinetic energy as time moves forward. This implies that the integrator is not reproducing the exactly circular trajectory, but instead introduces a small error in the form of a numerical reduction in the perpendicular velocity at every timestep. As before, this is not a physical property, but a numerical property brought on by the intergrator we chose. Note that for relatively short time periods this is not an issue and the RK4 provides a very good solution. A decreasing perpendicular velocity will only be apparent in the RK4 solutions if we take thousands of iterations.

1.4 Results

We will be solving the Lorentz force equation for a charged particle in a specific electromagnetic field configuration. The initial conditions are:

- $\mathbf{x}_0 = \langle 0, 0, 0 \rangle$ [m],
- $\mathbf{v}_0 = \langle 1, 0, 0 \rangle$ [m/s],
- $\mathbf{E} = \langle 0, 0, 0 \rangle$ [N/C],
- $\mathbf{B} = \langle 0, 0, 1 \rangle$ [T].

We assume normalized units, i.e., $q = 1$ and $m = 1$. The timesteps we will use are $\Delta t = [0.1, 0.05, 0.01, 0.005]$ and the simulation will run over 4 cycles.

In Fig. 1.1, we show the exact and numerical solutions to the Lorentz force and the gyromotion for timestep $dt = 0.01$ using the Forward Euler method integrator. The top row shows the position, the middle row the velocity, and the bottom row the absolute error between the exact solution and the numerical solution from the Euler method. In Fig. 1.2, we show the same quantities, but for timestep $dt = 0.1$.

As discussed in Sec. 1.3, the Forward Euler method begins to lose accuracy as time goes on. This is evident from the plots on the left that show the position and velocity versus time. Initially, the position starts off well-approximated, but as more cycles are completed, the numerical velocity and position solutions begin to overshoot the exact solution. This is also evident by the plots on the right that show y vs. x and v_y vs. v_x. The particle begins to spiral out of the expected circular trajectory. As the timestep increases, the inaccuracies greatly increase and the particle completely leave the exact trajectory. If we look at the bottom row of plots, the signed residual oscillates and the amplitudes of the oscillations slowly start growing as the particle travels more cycles.

In Fig. 1.3, we show the absolute, relative, and residual error of the kinetic energy calculated using the Forward Euler method integrator for both timesteps: (top) $dt = 0.01$ and (bottom) $dt = 0.1$. The residual error is signed, as opposed to the absolute and relative errors. As discussed in Sec. 1.3, the kinetic energy increases as times goes on. For the smaller timestep (top plots), the kinetic energy appears linear. For the larger timestep (bottom plots), the kinetic energy now takes an increasing exponential form. This is exactly what we derived from the Forward Euler method integrator. The absolute and relative errors show the same conclusion.

In Fig. 1.4, we show the exact and numerical solutions to the Lorentz force and the gyromotion for timestep $dt = 0.01$ using the RK4 integrator. The top row shows the position, the middle row the velocity, and the bottom row the absolute error between the exact solution and the numerical solution from the RK4 method. In Fig. 1.5, we show the same quantities, but for timestep $dt = 0.1$.

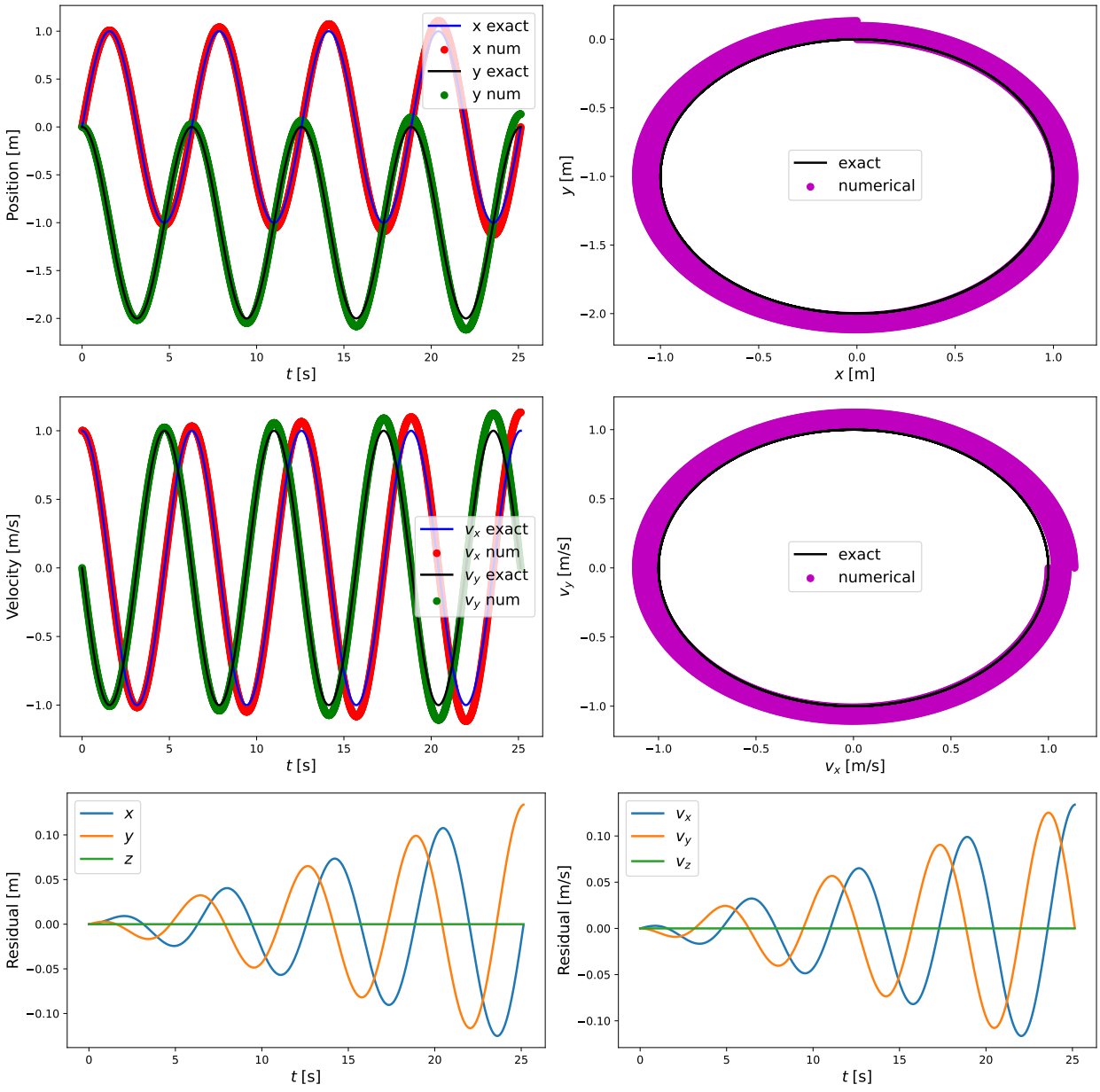


Figure 1.1: Euler method solutions with errors with timestep $dt = 0.01$.

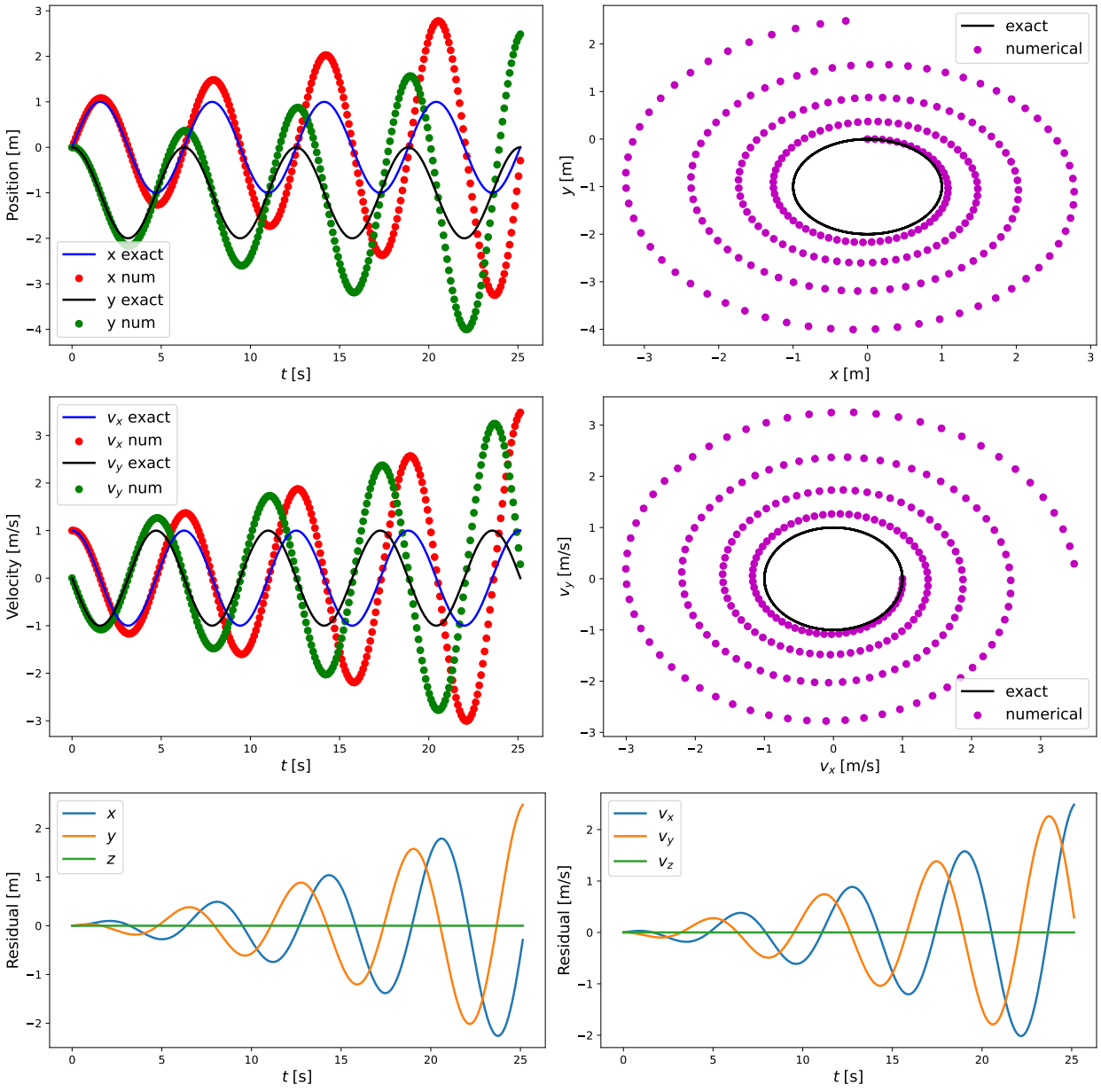


Figure 1.2: Euler method solutions with errors with timestep $dt = 0.1$.

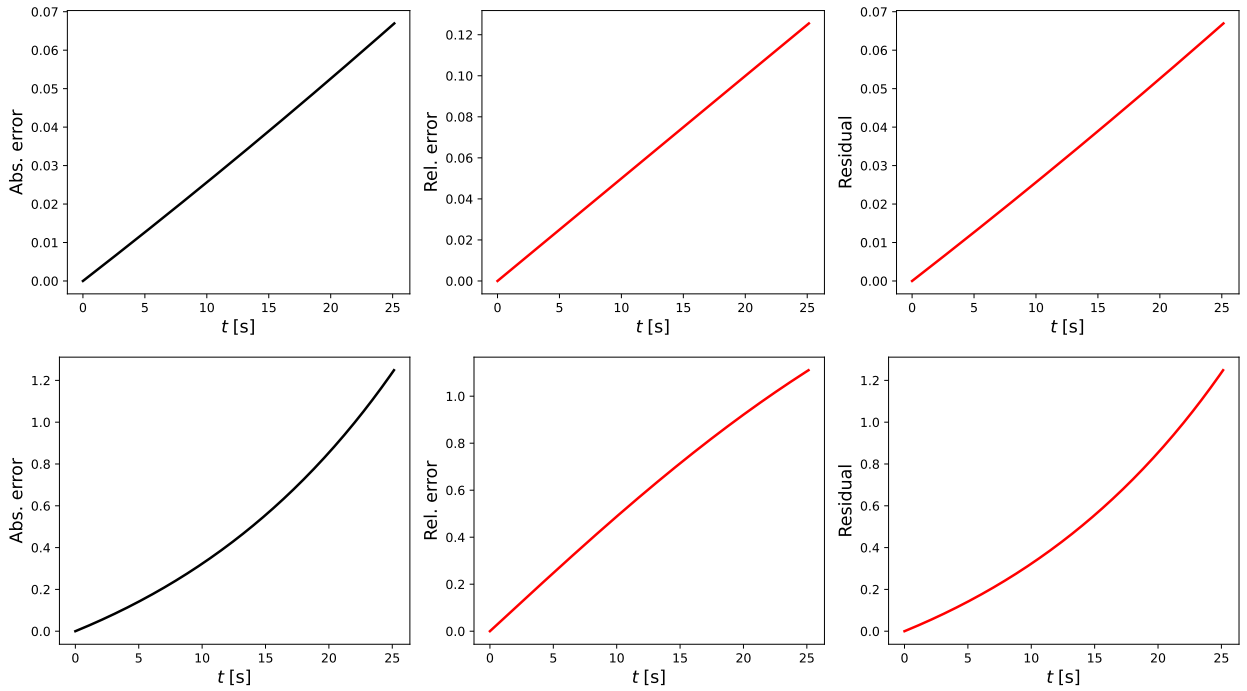


Figure 1.3: Kinetic energy calculated using the Euler method solutions for two different timesteps: (top) $dt = 0.01$ and (bottom) $dt = 0.1$.

As discussed in Sec. 1.3, the RK4 method provides stable solutions for our system throughout the time period we chose. The absolute errors are on order of 10^{-9} for timestep $dt = 0.01$. For the timestep $dt = 0.1$, the absolute errors do fall to 10^{-5}. Although the solutions are still fairly accurate, this decrease in accuracy is exactly what we predicted in Sec. 1.3 as the timestep increases.

In Fig. 1.6, we show the absolute, relative, and residual error of the kinetic energy calculated using the RK4 method integrator for both timesteps: (top) $dt = 0.01$ and (bottom) $dt = 0.1$. As previously mentioned, the residual error is signed, as opposed to the absolute and relative errors. From the plots, we see that the kinetic energy does indeed decrease with consecutive time steps. As we predicted, the accuracy decreases as we increase the timestep. The error for the timestep $dt = 0.01$ is on the order of 10^{-12}, while the error for the timestep $dt = 0.1$ is on the order of 10^{-7}. This shows that the RK4 provides much more accurate solutions compared to the Forward Euler method.

In Fig. 1.7, we show the convergence of the different integrators. For comparison purposes, we also included an additional integrator known as the Heun method. This is a second-order method that improves upon the Euler method and can be derived from the general RK4 method. The figure plots the natural logarithm of the maximum positional error versus the natural logarithm of the timestep size to determine the specified numerical method's order of convergence. We have also drawn a best-fit line.

In general, the slope of the best-fit line in log-log space should be ~ 1 for a first-order method,

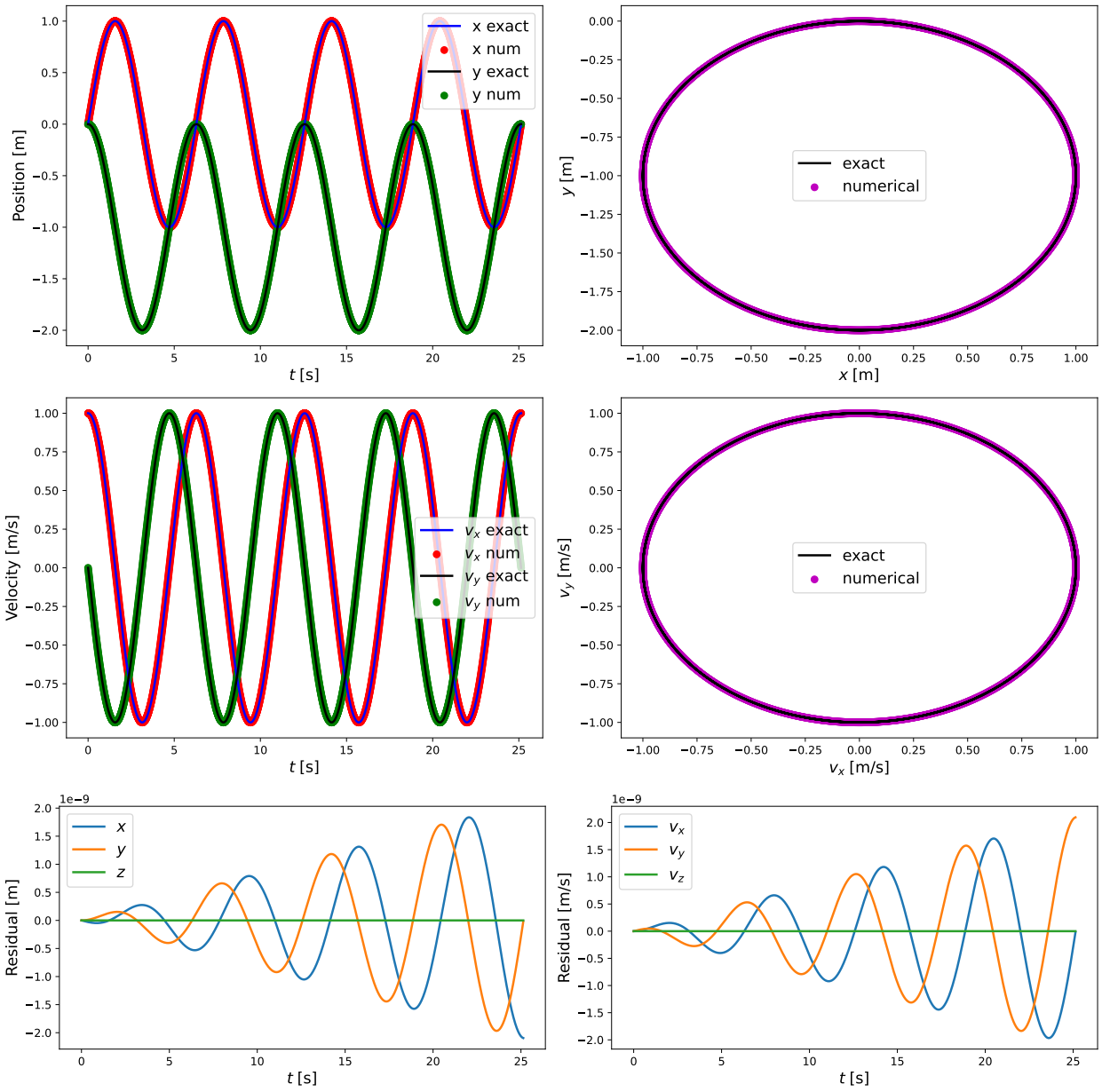


Figure 1.4: RK4 method solutions with errors with timestep $dt = 0.01$.

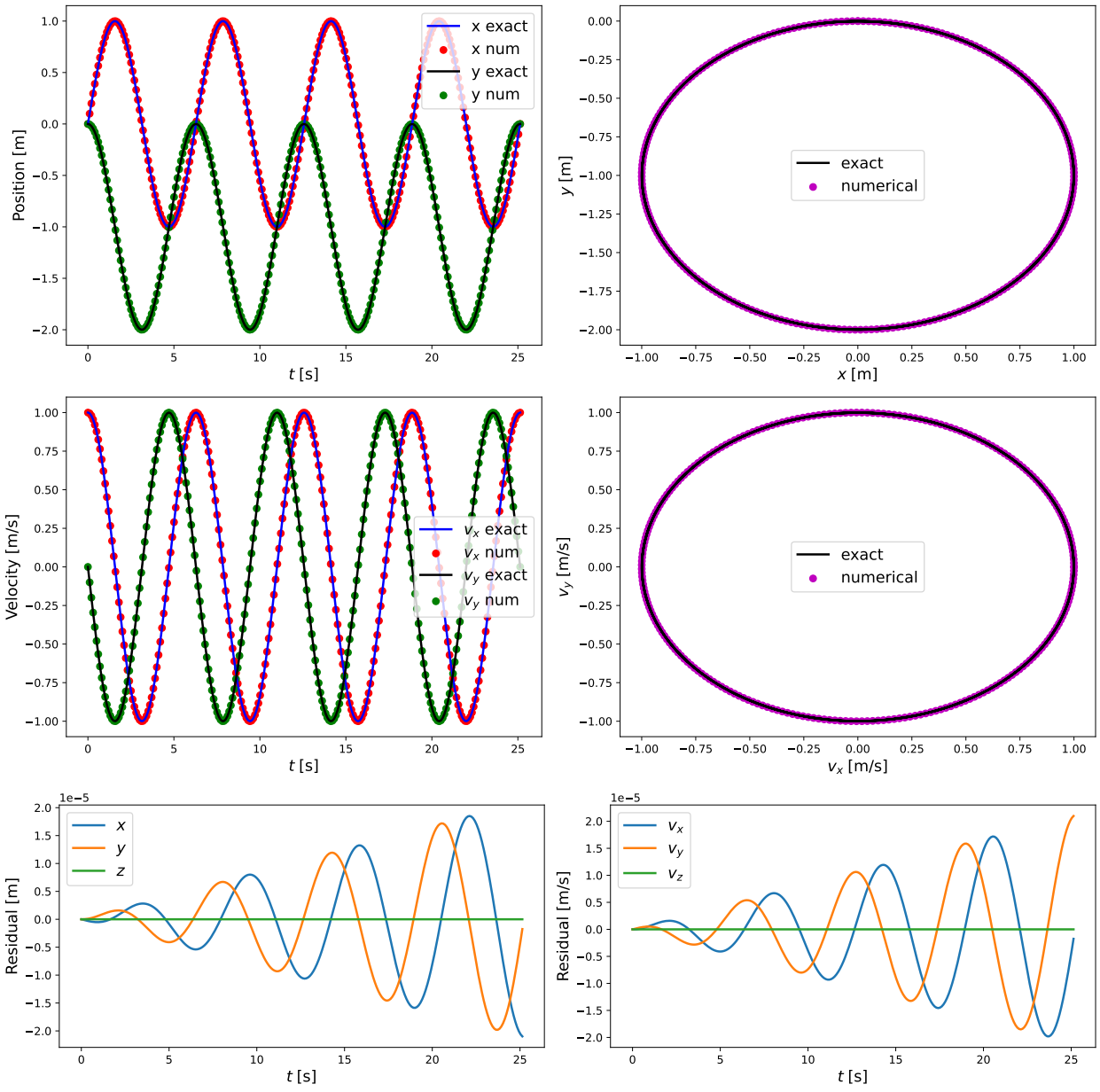


Figure 1.5: RK4 method solutions with errors with timestep $dt = 0.1$.

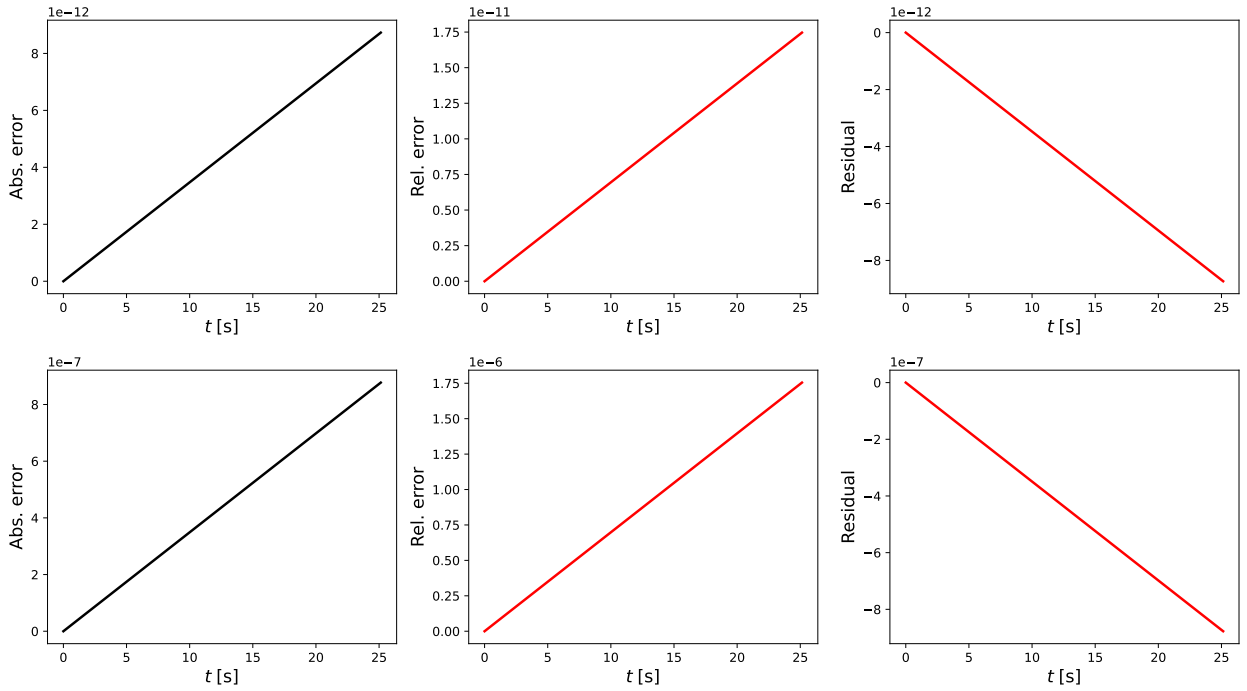


Figure 1.6: Kinetic energy calculated using the RK4 method solutions for two different timesteps: (top) $dt = 0.01$ and (bottom) $dt = 0.1$.

~ 2 for a second-order method, and ~ 4 for a fourth-order method. From the figure, we see that the best-fit lines for the Heun and RK4 methods do in fact have the appropriate slopes, which we denote as p. On the other hand, the slope of the best-fit line for Euler's method is not quite 1, but this is due to the inaccuracies introduced due to the timestep size and the number of cycles as we discussed in Sec. 1.3 and as the error plots have shown. Overall, these results confirm that our integrators work as expected.

1.5 Conceptual Questions

1. **Question:**
   Why does a magnetic field change velocity direction but not particle kinetic energy?

   Answer:
   The kinetic energy is dependent on the speed of the particle, not the direction. Thus, the kinetic energy will not change as long as the magnitude does not change.

2. **Question:**
   What determines the sign of cyclotron rotation?

   Answer:
   The sign is determined by the charge of the particle.

3. **Question:**
   What determines the Larmor radius?

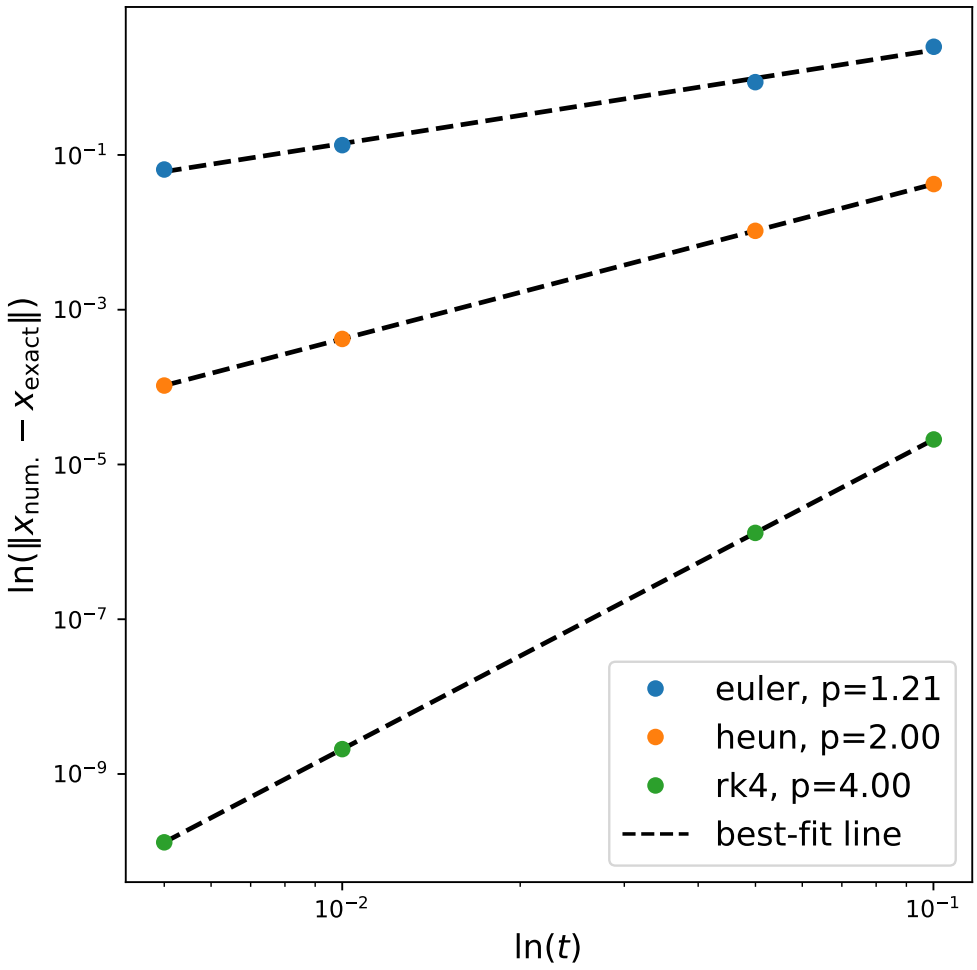


Figure 1.7: Integrator convergence.

Answer:
The Larmor radius is determined by the perpendicular velocity and the cyclotron frequency.

4. **Question:**
What happens to ρ_L if B doubles?

Answer:
Since ρ_L has an inverse proportional relationship to B through the cyclotron frequency, ρ_L will decrease by a factor of 2.

5. **Question:**
Why does $v_{\parallel}$ remain constant in a uniform magnetic field?

Answer:
The parallel velocity will remain constant since it is orthogonal to the direction of the magnetic field.

6. **Question:**
   Why does Euler's method artificially increase particle energy?

   Answer:
   Since Euler's method is first-order, error starts to build up as the timesteps change, thus increasing the energy.

7. **Question:**
   What does numerical convergence mean?

   Answer:
   Numerical convergence refers to how a numerical approximation approaches the exact solution as the numerical resolution is increased.

8. **Question:**
   Why isn't a visually correct orbit sufficient validation?

   Answer:
   It is not enough to determine whether or not the solutions are accurate since small deviations can occur due to numerical error. Additionally, looking at the orbit does not tell you if the charge has the correct sign.

9. **Question:**
   What physical timescale determines an appropriate Δt?

   Answer:
   The period, T_c, depends on Δt.

10. **Question:**
    Why are normalized units useful?

    Answer:
    Normalized units are useful for avoiding issues with varying magnitudes and makes interpreting the results easier.

1.6 Summary

In this chapter, we went over the derivation of the dynamics of a charged particle in a magnetic field. We showed the general solutions under two limiting cases, in what scenarios the energy is conserved, and how the equations can be written in terms of the parallel and perpendicular velocity components.

We also introduced two different integrators for solving differential equations and explored some of their properties. With these integrators, we solved the Lorentz force over different timesteps for a specific electromagnetic field configuration and analyzed the results. Finally, we plotted the convergence of the different integrators to verify that they were working correctly.

Overall, we verified that our simulation accurately reproduced the physical behavior predicted by our theoretical derivations. We also confirmed that the integrators exhibited the

expected properties. These results provide a solid foundation for applying these numerical methods to more complex physical systems in future simulations.

Appendix A

Levi-Civita symbol

Let's briefly review the Levi-Civita symbol. It is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k = 1,2,3 \text{ or } 2,3,1 \text{ or } 3,1,2 \text{ (even permutations)} \\ -1 & \text{if } i,j,k = 3,2,1 \text{ or } 2,1,3 \text{ or } 1,3,2 \text{ (odd permutations)} \\ 0 & \text{if } i = j \text{ or } i = k \text{ or } j = k. \end{cases} \tag{A.1}$$

The Levi-Civita symbol describes the permutations used in operations like cross products. As such, if we perform the cross product operation with two Levi-Civita symbols and apply the definition of the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{A.2}$$

we obtain the following relationship:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{jm}(\delta_{ii}\delta_{kn} - \delta_{in}\delta_{ki}) + \delta_{kn}(\delta_{ii}\delta_{jm} - \delta_{im}\delta_{ji})$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.3}$$

Other relationships can be derived such as

$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \tag{A.4}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{A.5}$$

References

[1] Francis F. Chen. *Introduction to Plasma Physics and Controlled Fusion*. 3rd ed. Cham, Switzerland: Springer, 2016. ISBN: 978-3-319-22308-7. DOI: 10.1007/978-3-319-22309-4.